GENERATING FUNCTIONS OF CHEBYSHEV-LIKE POLYNOMIALS

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Abstract. In this short note, we give simple proofs of several results and conjectures formulated by Stolarsky and Tran concerning generating functions of some families of Chebyshev-like polynomials.

1. Introduction

Stolarsky [5] observed that the discriminant of the product of the polynomials

$$K_s(x,y) = (1+y)^{2s} + xy^s \quad \text{and} \quad f_m(y) = (y^{2m+1}-1)/(y-1)$$

has a very nice factorization:

$$\Delta_y(K_s f_m) = C_m^{(s)} x^{2s-1}(x+2^{2s}) H_m^{(s)}(x)^4, \tag{1.1}$$

with $C_m^{(s)}$ a rational constant and $H_m^{(s)}(x)$ the following polynomial in $\mathbb{Q}[x]$:

$$H_m^{(s)}(x) = \prod_{k=1}^{m} \left(x + 4^s \cos^{2s}\left(\frac{k\pi}{2m+1}\right)\right). \tag{1.2}$$

This was obtained empirically for specific values of m and s. Moreover, Stolarsky conjectured that the generating functions of the polynomials $H_m^{(s)}$ for fixed positive integer s are rational, with explicit empirical formulas for $s = 1, 2, 3$. In [6], Tran proved Eq. (1.1) with the explicit value $C_m^{(s)} = (-1)^m (2m+1)^{2m-1} s^{2s}$ and obtained the generating function of the sequences of polynomials $H_m^{(1)}(x)$ and $H_m^{(2)}(x)$.

Indeed, for $s = 1$, the polynomials $H_m^{(s)}(x)$ are related to the classical Chebyshev polynomials of the second kind $U_n(x)$. From the definition

$$U_n(x) = 2^n \cdot \prod_{k=1}^{n} \left(x - \cos \frac{k\pi}{n+1} \right),$$

it follows that $U_{2m}(x) = (-1)^m \cdot H_m^{(1)}(-4x^2)$. From there, a simple derivation gives the generating function of the polynomials $H_m^{(1)}(x)$.

Proposition 1. *The generating function of the sequence $H_m^{(1)}(x)$ is rational:*

$$\sum_{m \geq 0} H_m^{(1)}(x) t^m = \frac{1-t}{(1-t)^2 - xt}.$$

2000 *Mathematics Subject Classification.* 11C08.

Work of the first two authors was supported in part by the Microsoft Research-Inria Joint Centre.

The last author was supported by National Science Foundation grant DMS 08-38434 EMSW21-MCTP: Research Experience for Graduate Students.

Proof. Starting from the classical rational generating function

$$\sum_{n\geq 0} U_n(x)t^n = \frac{1}{1-2xt+t^2}, \tag{1.3}$$

a generating function for the even part is readily obtained:

$$\sum_{m\geq 0} U_{2m}(x)t^{2m} = \frac{1}{2}\left(\frac{1}{1-2xt+t^2}+\frac{1}{1+2xt+t^2}\right) = \frac{1+t^2}{(1+t^2)^2-4x^2t^2},$$

and thus we deduce

$$\sum_{m\geq 0} H_m^{(1)}(-4x^2)t^m = \sum_{m\geq 0} U_{2m}(x)(-t)^m = \frac{1-t}{(1-t)^2+4x^2t},$$

from which the conclusion follows. □

In [6], Proposition 1 was proved in a different way, and the generating function for the case $s=2$ was shown to be rational. We generalize these results by proving:

Theorem 1. *For any $s\geq 1$, the generating function $F_s(x,t) = \sum_{m\geq 0} H_m^{(s)}(x)t^m$ belongs to $\mathbb{Q}(x,t)$. Its denominator has degree at most 2^s in t, and its numerator has degree at most 2^s-1 in t.*

Moreover, we give an algorithm that computes these rational functions explicitly. In Section 2, we prove this theorem. In Section 3, we comment on the computational aspects. We conclude with a few further observations in Section 4.

2. Rational Generating Series

For convenience, we work with the polynomial $G_m^{(s)}(x) = (-1)^m H_m^{(s)}(-x)$. It is monic, with roots the s-th powers of the roots of $G_m^{(1)}(x) = (-1)^m H_m^{(1)}(-x)$. To prove the theorem, it is clearly sufficient to show that the generating function $\sum_{m\geq 0} G_m^{(s)}(x)t^m$ is rational.

2.1. **Roots.**

Letting ε_s be a primitive s-th root of unity, we have

$$G_m^{(s)}(x^s) = \prod_{G_m^{(1)}(\alpha)=0} (x^s-\alpha^s) = \prod_{G_m^{(1)}(\alpha)=0}\left(\prod_{j=0}^{s-1}(x-\alpha/\varepsilon_s^j)\right),$$

which equals

$$\prod_{j=0}^{s-1}\left(\prod_{G_m^{(1)}(\alpha)=0}(x-\alpha/\varepsilon_s^j)\right) = \prod_{j=0}^{s-1}\varepsilon_s^{-jm}\cdot\prod_{j=0}^{s-1}G_m^{(1)}(\varepsilon_s^j x) = (-1)^{m(s-1)}\cdot\prod_{j=0}^{s-1}G_m^{(1)}(\varepsilon_s^j x).$$

Therefore, the polynomial $G_m^{(s)}$ can be expressed using solely $G_m^{(1)}$ by

$$G_m^{(s)}(x^s) = (-1)^{m(s-1)}\cdot\prod_{j=0}^{s-1}G_m^{(1)}(\varepsilon_s^j x), \tag{2.1}$$

and in particular it is given by the resultant $G_m^{(s)}(x) = \mathrm{Res}_y\big(G_m^{(1)}(y), x-y^s\big)$.

2.2. **Polynomials, Recurrences and Hadamard Products.** A consequence of Proposition 1 is that the sequence of polynomials $G_m^{(1)}(x)$ satisfies the linear recurrence

$$G_{m+2}^{(1)}(x) + (2-x)G_{m+1}^{(1)}(x) + G_m^{(1)}(x) = 0, \quad G_0(x) = 1,\ G_1(x) = x-1. \tag{2.2}$$

The coefficients of this recurrence are *constant*, in the sense that they do not depend on the index m. The characteristic polynomial of the sequence $G_m^{(1)}(x)$ is the reciprocal $t^2 + (2-x)t + 1$ of the denominator of its generating function.

Conversely, any sequence satisfying a linear recurrence with constant coefficients admits a rational generating function and this applies in particular to the sequence of polynomials $G_m^{(1)}(\varepsilon_s^j x)$ for any $j \geq 0$.

The product of sequences u_n and v_n that are solutions of linear recurrences with constant coefficients satisfies a linear recurrence with constant coefficients again (see, e.g., [7, §2.4], [4, Prop. 4.2.5], [2, §2, Ex. 5]). Its generating function, called the *Hadamard product* of those of u_n and v_n, is therefore rational. From Eq. (2.1), it follows that the generating function $\sum_{m\geq 0} G_m^{(s)}(x^s)t^m$ is also rational (*a priori* belonging to $\mathbb{C}(x,t)$).

Moreover, the reciprocal of the denominator of a Hadamard product of rational series has for roots the pairwise products of those of the individual series. Thus, letting $\alpha_1(x)$, $\alpha_2(x)$ denote the roots of $(1+t)^2 - xt$, the reciprocal of the denominator of the generating function of $(-1)^{m(s-1)} \cdot G_m^{(s)}(x^s)$ is the characteristic polynomial

$$P_s(x,t) = \prod_{1\leq i_1,\dots,i_s\leq 2} \Big(t - \alpha_{i_1}(x)\alpha_{i_2}(\varepsilon_s x)\cdots\alpha_{i_s}(\varepsilon_s^{s-1}x)\Big).$$

Note that, since $\alpha_1 \cdot \alpha_2 = 1$, the polynomial P_s is self-reciprocal with respect to t.

We prove that the polynomial $P_s(x,t)$ belongs to $\mathbb{Q}[x^s,t]$ by showing that all the (Newton) powersums of the roots of $P_s(x,t)$ belong to $\mathbb{Q}[x^s]$. For any $\ell \in \mathbb{N}$, the ℓ-th powersum is equal to the product

$$\left(\alpha_1^\ell(x) + \alpha_2^\ell(x)\right)\left(\alpha_1^\ell(\varepsilon_s x) + \alpha_2^\ell(\varepsilon_s x)\right)\cdots\left(\alpha_1^\ell(\varepsilon_s^{s-1}x) + \alpha_2^\ell(\varepsilon_s^{s-1}x)\right), \tag{2.3}$$

and thus has the form $T_{\ell,s}(x) = Q_\ell(x)Q_\ell(\varepsilon_s x)\cdots Q_\ell(\varepsilon_s^{s-1}x)$ for some polynomial $Q_\ell(x) \in \mathbb{Q}[x]$. The polynomial $T_{\ell,s}(x)$ being left unchanged under replacing x by $\varepsilon_s x$, it belongs to $\mathbb{Q}[x^s]$, and so do all the coefficients of P_s. This can also be seen from the expression of $T_{\ell,s}$ as a resultant: $T_{\ell,s}(x) = \operatorname{Res}_y(y^s - x^s, Q_\ell(y))$.

In conclusion, $G_m^{(s)}(x^s)$ satisfies a recurrence with coefficients that are polynomials in $\mathbb{Q}[x^s]$, thus the series $\sum_{m\geq 0} G_m^{(s)}(x^s)t^m$ is rational and belongs to $\mathbb{Q}(x^s,t)$, and thus $F_s(x,t)$ belongs to $\mathbb{Q}(x,t)$. The assertion on the degree in t of (the denominator and numerator of) $F_s(x,t)$ follows from the form of $P_s(x,t)$. This concludes the proof of Theorem 1.

3. Algorithm

The proof of Theorem 1 is actually effective, and therefore it can be used to generate, for specific values of s, the corresponding rational function F_s, in a systematic and unified way.

3.1. **New Proof for the Case $s = 2$.** We illustrate these ideas in the simplest case $s = 2$, for which the computations can be done by hand.

We compute the generating function of $k_m(x) := H_m^{(2)}(-x^2) = G_m^{(1)}(x)G_m^{(1)}(-x)$ from which that of the sequence $H_m^{(2)}(x)$ is easily deduced. By Proposition 1, the generating functions of the sequences $G_m^{(1)}(x)$ and $G_m^{(1)}(-x)$ have for denominators

$$g_1(t) := (1+t)^2 - xt, \qquad g_2(t) := (1+t)^2 + xt,$$

which are self-reciprocal. The polynomial whose roots are the products of the roots of g_1 and g_2 is obtained by a resultant computation:

$$\operatorname{Res}_u(g_1(u), u^2 g_2(t/u)) = (t-1)^4 + x^2 t(1+t)^2 = 1 + (x^2-4)t + (2x^2+6)t^2 + (x^2-4)t^3 + t^4.$$

It follows that the sequence $k_m(x)$ satisfies the fourth order recurrence

$$k_{m+4} + (x^2-4)k_{m+3} + (2x^2+6)k_{m+2} + (x^2-4)k_{m+1} + k_m = 0.$$

The initial conditions can be determined separately

$$k_0 = 1,\ k_1 = 1 - x^2,\ k_2 = 1 - 7x^2 + x^4,\ k_3 = 1 - 26x^2 + 13x^4 - x^6$$

and used to compute the numerator of the generating function of the sequence of polynomials $H_m^{(2)}(x)$. In conclusion, we have just proven the identity

$$\sum_{m=0}^{\infty} H_m^{(2)}(x)t^m = \frac{(1-t)^3}{(t-1)^4 - xt(t+1)^2}. \tag{3.1}$$

This provides a simple alternative proof of [6, Prop. 4.1].

3.2. **General Algorithm.** The computation of Hadamard products extends to linear recurrences that can even have *polynomial* coefficients. It is implemented in the Maple package `gfun` [3]. This makes it possible to write a first implementation that produces the generating function of the $H_m^{(s)}(x)$ for small s.

The computation can be made more efficient by using recent algorithms specific to linear recurrences with constant coefficients [1] (or, equivalently, to rational functions). The central idea is to avoid the use of a general-purpose algorithm for the computation of bivariate resultants and recover the denominator from the (Newton) powersums of its roots, themselves obtained from the powersums of α_1 and α_2. Using very simple univariate resultants, computations in algebraic extensions can be avoided. This is summarized in the following algorithm that follows the steps of the constructive proof of Thm. 1, using efficient algorithmic tools.

Input: an integer $s \geq 2$.
Output: the rational function $F_s(x,t)$.

(1) Compute the powersums $Q_\ell(x) := \alpha_1(x)^\ell + \alpha_2(x)^\ell \in \mathbb{Q}[x]$ for $\ell = 0, \ldots, 2^s$;
(2) Compute the powersums $T_{\ell,s}$ from Eq. (2.3) by $T_{\ell,s}(x^{1/s}) = \operatorname{Res}_y(y^s - x, Q_\ell(y))$ for $\ell = 0, \ldots, 2^s$;
(3) Recover the polynomial $P_s(x^{1/s}, t)$ from its powersums $T_{\ell,s}(x^{1/s}, t)$;
(4) Deduce the denominator $D_s(x,t)$ of $F_s(x,t)$ using $P_s(x^{1/s}, t) = D_s(-x, (-1)^s t)$;
(5) Compute $G_m^{(1)}(x)$ for $m = 0, \ldots, 2^s - 1$ using the 2nd order recurrence (2.2);
(6) Compute $G_m^{(s)}(x) = \operatorname{Res}_y(G_m^{(1)}(y), x - y^s)$ for $m = 0, \ldots, 2^s - 1$;
(7) Compute the numerator $N_s(x,t) := D_s(x,t) \times \sum_{m=0}^{2^s-1} G_m^{(s)}(-x)(-t)^m \bmod t^{2^s}$;
(8) **Return** the rational function $F_s(x,t) = N_s(x,t)/D_s(x,t)$.

Steps (1) and (3) can be performed efficiently using the algorithms in [1]. In Step (4) we use the fact that $P_s(x,t)$ is the denominator of the generating series

$$\sum_{m\geq 0}(-1)^{m(s-1)}\cdot G_m^{(s)}(x^s)t^m=\sum_{m\geq 0}H_m^{(s)}(-x^s)\big((-1)^s t\big)^m=F_s(-x^s,(-1)^s t).$$

We give the complete (and remarkably short) Maple code in an Appendix.

3.3. **Special Cases.** For $s=2$, the computation recovers (3.1). For $s=3$, it takes less than one hundredth of a second of computation on a personal laptop to prove the following result

$$\sum_{m\geq 0}H_m^{(3)}(x)t^m=\frac{(1-t)\left((t-1)^6-xt^2(t+3)(3t+1)\right)}{x^2t^4-xt(t^4+14t^3+34t^2+14t+1)(t-1)^2+(t-1)^8},$$

that was conjectured by Stolarsky in [5].

Similarly, less than two hundredth of a second of computation is enough to discover and prove the following new result.

Proposition 2. *The generating function* $\sum_{m\geq 0}H_m^{(4)}(x)t^m$ *is equal to*

$$\frac{(t-1)\left(x^2t^4A(t)-2xt^2\,B(t)\,(t-1)^6+(t-1)^{14}\right)}{x^3t^5(t+1)^2\,(t-1)^4+x^2t^3\,C(t)+xt(t-1)^8\,D(t)\,-(t-1)^{16}},$$

where

$$\begin{aligned}
A(t)&=9t^6-46t^5-89t^4-260t^3-89t^2-46t+9,\\
B(t)&=11t^4+128t^3+266t^2+128t+11,\\
C(t)&=2t^{10}-13t^9+226t^8-300t^7-676t^6-2574t^5-676t^4-300t^3+226t^2-13t+2,\\
D(t)&=t^6+60t^5+519t^4+1016t^3+519t^2+60t+1.
\end{aligned}$$

The code given in the appendix makes it possible to compute all the rational functions $F_s(x,t)$ for $1\leq s\leq 7$ in less than a minute.

4. Further Remarks

While computing with the polynomials $H_m^{(s)}$ we experimentally discovered the following amusing facts.

Fact 1. *For all* $s\geq 1$*, we have:*

$$\begin{aligned}
H_0^{(s)}(x)&=1,\\
H_1^{(s)}(x)&=x+1,\\
H_2^{(s)}(x)&=x^2+L_{2s}\,x+1,
\end{aligned}$$

where L_n *denotes the* n*-th element of the Lucas sequence defined by* $L_0=2, L_1=1$ *and* $L_n=L_{n-1}+L_{n-2}$ *for all* $n\geq 2$.

This is easy to prove.

Fact 2. $H_m^{(s)}(x)$ *has only non-negative integer coefficients, for all* $m\geq 0$ *and* $s\geq 1$.

Non-negativity is clear from (1.2), while the integrality follows from the generating function when $s = 1$ and from the resultant representation of $H_m^{(s)}$ for higher s.

Fact 2 suggests that the coefficients of the polynomials $H_m^{(s)}(x)$ could admit a nice combinatorial interpretation.

For instance, the coefficient of x^1 in $H_m^{(s)}(x)$ is equal to the trace of the matrix M^{2s}, where $M = (a_{i,j})_{i,j=1}^m$ is the $m \times m$ matrix with $a_{i,j} = 1$ for $i + j \le m + 1$ and $a_{i,j} = 0$ otherwise.

Fact 3. *The rational function $F_s(x,t) = \sum_{m\ge 0} H_m^{(s)}(x)t^m$ writes $N_s(x,t)/D_s(x,t)$, where $N_s(x,t)$ is a polynomial in $\mathbb{Q}[x,t]$ of degree $2^s - 1$ in t and $B_{s-1} - 1$ in x, $D_s(x,t)$ is a polynomial in $\mathbb{Q}[x,t]$ of degree 2^s in t and B_{s-1} in x, and where B_n is the central binomial coefficient*

$$B_n = \binom{n}{\lfloor n/2 \rfloor}.$$

Here is the sketch of a proof. We only prove upper bounds, but a finer analysis could lead to the exact degrees.

First, it is enough to show that the degree in x of the polynomial $P_s(x,t)$ defined in the proof of Theorem 1 is at most sB_{s-1}. Now, instead of considering P_s, we study the simpler polynomial

$$C_s(x,t) = \prod_{1\le i_1,\dots,i_s\le 2} \Big(t - \alpha_{i_1}(x)\alpha_{i_2}(x)\cdots\alpha_{i_s}(x)\Big).$$

We will prove the bound sB_{s-1} on the degree of C_s in x. Adapting the argument to the case of P_s is not difficult.

The starting point is that, when x tends to infinity, $\alpha_1(x)$ grows like x, while $\alpha_2(x)$ grows like x^{-1}. Therefore,

$$C_s(x,t) = \prod_{k=0}^{s} \Big(t - \alpha_1^{s-k}(x)\alpha_2^k(x)\Big)^{\binom{s}{k}}$$

grows like $L_s(x,t) = \prod_{k=0}^{s} \Big(t - x^{s-k}(x^{-1})^k\Big)^{\binom{s}{k}}$. The degree in x of $C_s(x,t)$ is thus bounded by the degree in x of the Laurent polynomial $L_s(x,t)$, which is equal to

$$\deg_x(L_s) = \sum_{k=0}^{\lfloor s/2\rfloor} \binom{s}{k}(s-2k) = \binom{s}{\lfloor s/2\rfloor + 1}\cdot\big(\lfloor s/2\rfloor + 1\big) = sB_{s-1}.$$

Appendix: Maple Code

For completeness, we give a self-contained Maple implementation that produces the rational form of the generating function $F_s(x,t)$.

```
recipoly:=proc(p,t) expand(t^degree(p,t)*subs(t=1/t,p)) end:

newtonsums:=proc(p,t,ord)
local pol, dpol;
  pol:=p/lcoeff(p,t);
  pol:=recipoly(pol,t);
  dpol:=recipoly(diff(pol,t),t);
```

```
  map(expand,series(dpol/pol,t,ord))
end:

invnewtonsums:=proc(S,t,ord) series(exp(Int((coeff(S,t,0)-S)/t,t)),t,ord) end:

Fs:=proc(s,x,t)
local H, G, N, Q, T, ell, y, Ps, Ds, G1, Gs, m, Ns;
  H:=(1-t)/((1-t)^2-x*t);
  G:=subs(x=-x,t=-t,H);
  N:=2^s;
  Q:=subs(x=y,newtonsums(denom(G),t,N+1));
  T:=series(add(resultant(y^s-x,coeff(Q,t,ell),y)*t^ell,ell=0..N),t,N+1);
  Ps:=convert(invnewtonsums(T,t,N+1),polynom);
  Ds:=subs(t=(-1)^s*t,x=-x,Ps); # denominator
  G1:=subs(x=y,series(G,t,N));
  Gs:=series(add(resultant(coeff(G1,t,m),x-y^s,y)*t^m,m=0..N),t,N);
  Ns:=convert(series(Ds*subs(x=-x,t=-t,Gs),t,N),polynom); # numerator
  collect(Ns/Ds,x,factor)
end:
```

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